

Chapter 6 Linear system

$$A\{x\} = \{b\} \quad \text{or} \quad A\bar{x} = \bar{b}$$

$$\begin{bmatrix} a_{11} & \bullet & \bullet & a_{1n} \\ \bullet & a_{22} & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ a_{n1} & \bullet & \bullet & a_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ \bullet \\ \bullet \\ x_n \end{Bmatrix} = \begin{Bmatrix} b_1 \\ \bullet \\ \bullet \\ b_n \end{Bmatrix}$$

§ Gaussian elimination

$$1. \text{ } i\text{th row} = i\text{th row} - \frac{a_{i1}}{a_{11}} \times 1^{\text{st}} \text{ row}, \quad i = 2 \sim n$$

$$2. \text{ } i\text{th row} = i\text{th row} - \frac{a_{i2}}{a_{22}} \times 1^{\text{st}} \text{ row}, \quad i = 3 \sim n$$

3. etc.

$$4. \text{ } n\text{th row} = n\text{th row} - \frac{a_{n,n-1}}{a_{n-1,n-1}} \times (n-1)\text{th row}$$

$$\begin{bmatrix} a_{11} & \bullet & \bullet & a_{1n} \\ 0 & a_{22} & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet \\ 0 & 0 & 0 & a_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ \bullet \\ \bullet \\ x_n \end{Bmatrix} = \begin{Bmatrix} c_1 \\ \bullet \\ \bullet \\ c_n \end{Bmatrix}$$

Backward substitution

$$x_n = \frac{c_n}{a_{nn}}, \quad x_i = \frac{1}{a_{ii}} \left[c_i - \sum_{j=i+1}^n a_{ij} x_j \right], \quad i = n-1 \sim 1.$$

§ Pivoting method

$$1. \text{ index max } \{|a_{11}|, \dots, |a_{n1}|\} = k$$

interchange the 1st row with the k th row.

$$i\text{th row} = i\text{th row} - \frac{a_{i1}}{a_{11}} \times 1^{\text{st}} \text{ row}, \quad i = 2 \sim n$$

$$2. \text{ index max } \{|a_{22}|, \dots, |a_{n2}|\} = k$$

interchange the 2nd row with the k th row.

$$\text{ith row} = \text{ith row} - \frac{a_{i2}}{a_{22}} \times 1^{\text{st}} \text{ row}, \quad i = 3 \sim n$$

3. etc.

4. performing the procedures of backward substitution.

§ Decomposition method

Decompose $A = LU$ in which

$$L = \begin{bmatrix} l_{11} & 0 & 0 & 0 \\ \bullet & l_{22} & 0 & 0 \\ \bullet & \bullet & \bullet & 0 \\ l_{1n} & \bullet & \bullet & l_{nn} \end{bmatrix}, \quad U = \begin{bmatrix} u_{11} & \bullet & \bullet & u_{1n} \\ 0 & u_{22} & \bullet & u_{2n} \\ 0 & 0 & \bullet & \bullet \\ 0 & 0 & 0 & u_{nn} \end{bmatrix}$$

Then

$$U\bar{x} = \bar{y}, \quad L\bar{y} = \bar{b},$$

$$\text{Where } y_1 = \frac{b_1}{l_{11}}, \quad y_i = \frac{1}{l_{ii}}[b_i - \sum_{j=1}^{i-1} l_{ij}y_j], \quad i = 2 \sim n.$$

$$x_n = \frac{y_n}{u_{nn}}, \quad x_i = \frac{1}{u_{ii}}[y_i - \sum_{j=i+1}^n u_{ij}x_j], \quad i = 2 \sim n.$$

Case 1. $l_{ii} = 1, \quad i = 1 \sim n$

$$a_{ij} = \sum_{k=1}^n l_{ik}u_{kj} = \sum_{k=1}^{i-1} l_{ik}u_{kj} + l_{ii}u_{ij} + \sum_{k=i+1}^n l_{ik}u_{kj} = \sum_{k=1}^{i-1} l_{ik}u_{kj} + u_{ij},$$

$$u_{ij} = a_{ij} - \sum_{k=1}^{i-1} l_{ik}u_{kj}, \quad i \leq j.$$

$$a_{kj} = \sum_{p=1}^n l_{kp}u_{pj} = \sum_{p=1}^{j-1} l_{kp}u_{pj} + l_{kj}u_{jj} + \sum_{p=j+1}^n l_{kp}u_{pj} = l_{kj}u_{jj} + \sum_{p=j+1}^n l_{kp}u_{pj}$$

$$l_{kj} = (a_{kj} - \sum_{p=j+1}^n l_{kp}u_{pj})/u_{jj}, \quad k \leq j.$$

Case 2. $u_{ii} = 1, \quad i = 1 \sim n$

$$a_{kj} = \sum_{p=1}^n l_{kp}u_{pj} = \sum_{p=1}^{j-1} l_{kp}u_{pj} + l_{kj}u_{jj} + \sum_{p=j+1}^n l_{kp}u_{pj} = l_{kj} + \sum_{p=j+1}^n l_{kp}u_{pj}$$

$$l_{kj} = a_{kj} - \sum_{p=j+1}^n l_{kp}u_{pj}, \quad k \leq j.$$

$$a_{ij} = \sum_{k=1}^n l_{ik} u_{kj} = \sum_{k=1}^{i-1} l_{ik} u_{kj} + l_{ii} u_{ij} + \sum_{k=i+1}^n l_{ik} u_{kj} = \sum_{k=1}^{i-1} l_{ik} u_{kj} + l_{ii} u_{ij},$$

$$u_{ij} = (a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj}) / l_{ii}, \quad i \leq j.$$

§ Positive definite matrix

$\forall \bar{x} \neq \bar{0}, \quad \bar{x}^T A \bar{x} > 0$: positive definite matrix

If A is positive and $A = A^T$, then A can be decomposed into the form

$A = LL^T$: Choleski's decomposition.

$$u_{ij} = l_{ji}, \quad i \geq j$$

$$a_{ii} = \sum_{k=1}^n l_{ik} u_{ki} = \sum_{k=1}^n l_{ik} l_{ik} = \sum_{k=1}^{i-1} l_{ik} l_{ik} + l_{ii} l_{ii}$$

$$l_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} l_{ik}^2}$$

$$a_{ij} = \sum_{k=1}^n l_{ik} u_{kj} = \sum_{k=1}^n l_{ik} l_{jk} = \sum_{k=1}^{i-1} l_{ik} l_{jk} + l_{ii} l_{ji} + \sum_{k=i+1}^n l_{ik} l_{jk} = \sum_{k=1}^{i-1} l_{ik} l_{jk} + l_{ii} l_{ji}, \quad i \leq j$$

$$l_{ji} = (a_{ij} - \sum_{k=1}^{i-1} l_{ik} l_{jk}) / l_{ii}$$

§ Band matrix

$$\text{a. } A = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{12} & \bullet & \bullet & 0 \\ 0 & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & a_{nn} \end{bmatrix} = A^T$$

Crout method

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{12} & \bullet & \bullet & 0 \\ 0 & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & a_{nn} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ 0 & \bullet & \bullet & 0 \\ 0 & 0 & l_{n,n-1} & l_{nn} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & 0 & 0 \\ 0 & 1 & \bullet & 0 \\ 0 & 0 & \bullet & u_{n-1,n} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$l_{11} = a_{11}, \quad a_{i,i-1} = l_{i,i-1} u_{i-1,i-1} + l_{i,i} u_{i,i-1} = l_{i,i-1},$$

$$a_{i,i} = l_{i,i-1} u_{i-1,i} + l_{i,i} u_{i,i} = l_{i,i-1} u_{i-1,i} + l_{i,i},$$

$$a_{i,i+1} = l_{i,i-1}u_{i-1,i+1} + l_{i,i}u_{i,i+1} + l_{i,i+1}u_{i+1,i+1} = l_{i,i}u_{i,i+1} + l_{i,i+1}$$

$$\text{b. } A = \begin{bmatrix} a_{11} & \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet & a_{nn} \end{bmatrix} = A^T$$

$$A = \begin{bmatrix} a_{11} & \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet & a_{nn} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 & 0 \\ l_{31} & \bullet & \bullet & 0 & 0 \\ 0 & \bullet & \bullet & \bullet & 0 \\ 0 & 0 & \bullet & \bullet & l_{nn} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} & 0 & 0 \\ 0 & 1 & u_{23} & \bullet & 0 \\ 0 & 0 & 1 & \bullet & \bullet \\ 0 & 0 & 0 & 1 & \bullet \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$l_{1j} = a_{1j}, \quad u_{12} = a_{12} / l_{11}, \quad u_{13} = a_{13} / l_{11},$$

$$a_{i,i} = l_{i,i-2}u_{i-2,i} + l_{i,i-1}u_{i-1,i} + l_{i,i}u_{i,i} = l_{i,i-2}u_{i-2,i} + l_{i,i-1}u_{i-1,i} + l_{i,i},$$

$$a_{i,i-1} = l_{i,i-2}u_{i-2,i-1} + l_{i,i-1}u_{i-1,i-1} + l_{i,i}u_{i,i-1} = l_{i,i-2}u_{i-2,i-1} + l_{i,i-1},$$

$$a_{i,i-2} = l_{i,i-2}u_{i-2,i-2} + l_{i,i-1}u_{i-1,i-2} + l_{i,i}u_{i,i-2} = l_{i,i-2}$$

$$a_{i,i+1} = l_{i,i-2}u_{i-2,i+1} + l_{i,i-1}u_{i-1,i+1} + l_{i,i}u_{i,i+1} = l_{i,i-1}u_{i-1,i+1} + l_{i,i}u_{i,i+1},$$

$$a_{i,i+2} = l_{i,i-2}u_{i-2,i+2} + l_{i,i-1}u_{i-1,i+2} + l_{i,i}u_{i,i+2} = l_{i,i}u_{i,i+2},$$

HW6

1. Use Gaussian elimination and pivoting technique to solve

$$1.19x_1 + 2.11x_2 - 100x_3 + x_4 = 1.12$$

$$14.2x_1 - 0.112x_2 + 12.2x_3 - x_4 = 3.44$$

$$100x_2 - 99.9x_3 + x_4 = 2.15$$

$$15.3x_1 + 0.110x_2 - 13.1x_3 - x_4 = 4.16$$

2. Find the inverse of the matrix A where

$$A = \begin{bmatrix} 4 & 1 & -1 & 0 \\ 1 & 3 & -1 & 0 \\ -1 & -1 & 6 & 2 \\ 0 & 0 & 2 & 5 \end{bmatrix}$$

3. Use Crout factorization for a tri-diagonal system to solve the problem

$$\begin{bmatrix} 3 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{Bmatrix}.$$